HOMOGENEOUS MATRICES

$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \end{array}\right]$ is a homogeneous matrix since all the constants are equal to 0.

Exactly one of the following is true for all homogeneous matrices:

1. It has only the trivial solution, i.e. $x = 0, y = 0, z = 0$ $(0, 0, 0)$.
2. It has an infinite number of solutions.

HETEROGENEOUS (INHOMOGENEOUS) MATRICES

$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \end{array}\right]$ is a heterogeneous (inhomogeneous) matrix since the constants are non-zero.

Exactly one of the following is true for all heterogeneous matrices:

1. It has only a unique (non-trivial) solution.
2. It has no solutions.
3. It has an infinite number of solutions.

SOLVING MATRICES THAT HAVE A UNIQUE SOLUTION

$\begin{cases} 3x - 8y - 6z = 6 \\ -2x + 2y + z = 4 \\ x + 4y + 2z = -2 \end{cases}$ (write the system in the matrix form) $\Leftrightarrow$

$\left[\begin{array}{ccc|c} \boxed{3} & -8 & -6 & 6 \\ -2 & 2 & 1 & 4 \\ 1 & 4 & 2 & -2 \end{array}\right]$ $R_1 \to R_3$ $(R_1\ becomes\ R_3)$ $R_3 \to R_1$ $(R_3\ becomes\ R_1)$ $\Leftrightarrow$

$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ \boxed{-2} & 2 & 1 & 4 \\ 3 & -8 & -6 & 6 \end{array}\right]$ $R_2 \to R_2 + 2R_1$ $(add\ 2R_1\ to\ R_2)$ $\Leftrightarrow$

$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 10 & 5 & 0 \\ \boxed{3} & -8 & -6 & 6 \end{array}\right]$ $R_3 \to R_3 - 3R_1$ $(subtract\ 3R_1\ from\ R_3)$ $\Leftrightarrow$

$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 10 & 5 & 0 \\ 0 & \boxed{-20} & -12 & 12 \end{array}\right]$ $R_3 \to R_3 + 2R_2$ $(add\ 2R_2\ to\ R_3)$ $\Leftrightarrow$

$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 10 & 5 & 0 \\ 0 & 0 & \boxed{-2} & 12 \end{array}\right]$ $R_3 \to R_3/-2$ $(divide\ R_3\ by - 2)$ $\Leftrightarrow$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & 10 & \boxed{5} & 0 \\ 0 & 0 & 1 & -6 \end{array}\right] R_2 \to R_2 - 5R_3 \qquad (subtract\ 5R_3\ from\ R_2) \qquad \Leftrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -2 \\ 0 & \boxed{10} & 0 & 30 \\ 0 & 0 & 1 & -6 \end{array}\right] R_2 \to R_2/10 \qquad (divide\ R_2\ by\ 10) \qquad \Leftrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & \boxed{2} & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -6 \end{array}\right] R_1 \to R_1 - 2R_3 \qquad (subtract\ 2R_3\ from\ R_1) \qquad \Leftrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & \boxed{4} & 0 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -6 \end{array}\right] R_1 \to R_1 - 4R_2 \qquad (subtract\ 4R_2\ from\ R_1) \qquad \Leftrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

Performing back substitution yields $\begin{cases} x = -2 \\ y = 3 \\ z = -6 \end{cases}$.

∴ The matrix has a unique solution $(-2, 3, 6)$.

SOLVING MATRICES THAT HAVE NO SOLUTION

$$\begin{cases} 6x - 2y + 3z = 4 \\ -5x + y + 7z = -3 \\ 8x - 4y + 23z = 8 \end{cases} \quad \text{(write the system in the matrix form)} \quad \Leftrightarrow$$

$$\left[\begin{array}{ccc|c} \boxed{6} & -2 & 3 & 4 \\ -5 & 1 & 7 & -3 \\ 8 & -4 & 23 & 8 \end{array}\right] R_1 \to R_1 + R_2 \qquad (add\ R_2\ to\ R_1) \qquad \Leftrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 10 & 1 \\ \boxed{-5} & 1 & 7 & -3 \\ 8 & -4 & 23 & 8 \end{array}\right] R_2 \to R_2 + 5R_1 \qquad (add\ 5R_1\ to\ R_2) \qquad \Leftrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 10 & 1 \\ 0 & -4 & 57 & 2 \\ \boxed{8} & -4 & 23 & 8 \end{array}\right] R_3 \to R_3 - 8R_1 \qquad (subtract\ 8R_1\ from\ R_3) \qquad \Leftrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 10 & 1 \\ 0 & -4 & 57 & 2 \\ 0 & \boxed{4} & -57 & 0 \end{array}\right] R_3 \to R_3 + R_2 \qquad (add\ R_2\ to\ R_3) \qquad \Leftrightarrow$$

$$\begin{bmatrix} 1 & -1 & 10 & | & 1 \\ 0 & -4 & 57 & | & 2 \\ 0 & 0 & 0 & | & 2 \end{bmatrix}$$

Performing back substitution for R_3 yields $0 = 2$. Clearly, this is impossible.

∴ The matrix has no solutions ($\emptyset$).

SOLVING MATRICES THAT HAVE AN INFINITE NUMBER OF SOLUTIONS

$$\begin{cases} -x - 3y - z = -9 \\ 3x + 4y - 2z = 7 \\ -2x - 3y + z = -6 \end{cases} \quad \text{(write the system in the matrix form)} \quad \Leftrightarrow$$

$$\begin{bmatrix} \boxed{-1} & -3 & -1 & | & -9 \\ 3 & 4 & -2 & | & 7 \\ -2 & -3 & 1 & | & -6 \end{bmatrix} R_1 \to -1R_1 \quad (multiply\ R_1\ by -1) \quad \Leftrightarrow$$

$$\begin{bmatrix} 1 & 3 & 1 & | & 9 \\ \boxed{3} & 4 & -2 & | & 7 \\ -2 & -3 & 1 & | & -6 \end{bmatrix} R_2 \to R_2 - 3R_3 \quad (subtract\ 3R_1\ from\ R_2)$$

$$\begin{bmatrix} 1 & 3 & 1 & | & 9 \\ 0 & 1 & -5 & | & -20 \\ \boxed{-2} & -3 & 1 & | & -6 \end{bmatrix} R_3 \to R_3 + 2R_1 \quad (add\ 2R_1\ to\ R_3) \quad \Leftrightarrow$$

$$\begin{bmatrix} 1 & 3 & 1 & | & 9 \\ 0 & \boxed{-5} & -5 & | & -20 \\ 0 & 3 & 3 & | & 12 \end{bmatrix} R_2 \to R_2 /-5 \quad (divide\ R_2\ by -5) \quad \Leftrightarrow$$

$$\begin{bmatrix} 1 & 3 & 1 & | & 9 \\ 0 & 1 & 1 & | & 4 \\ 0 & \boxed{3} & 3 & | & 12 \end{bmatrix} R_3 \to R_3 - 3R_2 \quad (subtract\ 3R_2\ from\ R_3) \quad \Leftrightarrow$$

$$\begin{bmatrix} 1 & 3 & 1 & | & 9 \\ 0 & 1 & 1 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

We have obtained a row of 0's in R_3. Performing back substitution yields:

$$\begin{cases} x + 3y + z = 9 \\ y + z = 4 \end{cases}$$ More variables than equations; therefore, let $z = t, t \in R$.

Substituting $z = t$ into the 2nd equation yields:

$$y + t = 4 \quad \Leftrightarrow \quad y = 4 - t$$

Substituting $y = 4 - t$ and $z = t$ into the 1st equation yields:

$$x + 3(4 - t) + t = 9 \quad \Leftrightarrow \quad x + 12 - 3t + t = 9 \quad \Leftrightarrow \quad x = 2t - 3$$

∴ The matrix has an infinite number of solutions $(2t - 3, 4 - t, t), t \in R$.

GAUSSIAN ELIMINATION: ROW ECHELON FORM (REF)

$\left[\begin{array}{ccc|c} 1 & b_1 & c_1 & k_1 \\ 0 & 1 & c_2 & k_2 \\ 0 & 0 & 1 & k_3 \end{array}\right]$ is said to be the in row echelon form.

Definition: The process of converting a matrix to the row echelon form is called the Gaussian Elimination (named after the mathematician Gauss).

Definition: A matrix is in the row echelon form if the following two conditions hold:

1. The leading 1's are distributed from left to right.
2. There are no rows of 0's between any leading 1's.
3. All entries below the leading 1's are equal to 0.

DECIDING WHETHER THE MATRIX IS IN REF

$\begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ Yes! The 3 conditions hold.

$\begin{bmatrix} 1 & 7 & -3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ No! The leading 1's are not distributed from left to right.

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ No! There is non-zero entry below the 2nd leading 1.

$\begin{bmatrix} 0 & 1 & 5 & 2 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ Yes! The 3 conditions hold.

$\begin{bmatrix} 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 3 \end{bmatrix}$ No! The leading 1's are not distributed from left to right.

$\begin{bmatrix} 1 & 7 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Yes! The 3 conditions hold.

$\begin{bmatrix} 1 & 7 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ No! There is a row of 0's between the leading 1's.

$\begin{bmatrix} 1 & 4 & 7 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Yes! The 3 conditions hold.

$[1]$ Yes! The 3 conditions hold.

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ No! There is no leading one in R_1.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Yes! The 3 conditions hold.

GAUSS-JORDAN ELIMINATION: ROW REDUCED ECHELON FORM (RREF)

$\begin{bmatrix} 1 & 0 & 0 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \end{bmatrix}$ is said to be in the reduced row echelon form.

Definition: The process of converting a matrix to the reduced row echelon form is called the Gauss-Jordan Elimination (named after the mathematicians Gauss and Jordan).

Definition: A matrix is in the reduced row echelon form if the following three conditions hold:

1. The leading 1's are distributed from left to right.
2. There are no rows of 0's between any leading 1's.
3. All entries below the leading 1's are equal to 0.
4. All entries above the leading 1's are equal to 0.

Note: Notice that if a matrix is in the RREF, it is also in the REF.

DECIDING WHETHER THE MATRIX IS IN RREF

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Yes! The 4 conditions hold.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ No! The leading 1's are not distributed from left to right.

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ No! There is non-zero entry above the 3rd leading 1.

$\begin{bmatrix} 0 & 1 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ Yes! The 4 conditions hold.

$\begin{bmatrix} 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 3 \end{bmatrix}$ No! The leading 1's are not distributed from left to right.

$\begin{bmatrix} 1 & 7 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Yes! The 4 conditions hold.

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ No! There is a row of 0's between the leading 1's.

$\begin{bmatrix} 1 & 9 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ Yes! The 4 conditions hold.

$[0]$ No! There is no leading 1 in R_1.

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ No! There is no leading 1 in R_1.

MATRIX SIZE

Definition: The size of a matrix is $(rows \times cols)$.

$[8]$ 1×1 $\begin{bmatrix} 3 & -2 \\ 11 & 41 \end{bmatrix}$ 2×2 $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -9 & -4 \end{bmatrix}$ 2×3 $\begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 4 & -5 \end{bmatrix}$ 3×2 $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ 1×3

SQUARE MATRIX

Definition: The matrix whose number of rows is equal to the number of columns.

$[8]$ 1×1 $\begin{bmatrix} 3 & -2 \\ 11 & 41 \end{bmatrix}$ 2×2 $\begin{bmatrix} 2 & 5 & 7 \\ 1 & 4 & 2 \\ 7 & 3 & 3 \end{bmatrix}$ 3×3 $\begin{bmatrix} 2 & 4 & 2 & 4 \\ 5 & 6 & 7 & 6 \\ 5 & 7 & 4 & 8 \\ 1 & 6 & 7 & 2 \end{bmatrix}$ 4×4

UPPER DIAGONAL / TRIANGULAR MATRIX

Definition: The matrix whose entries above the main diagonal are non-zero is upper diagonal.

$[8]$ 1×1 $\begin{bmatrix} 3 & -2 \\ 0 & 41 \end{bmatrix}$ 2×2 $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ 3×3 $\begin{bmatrix} 2 & 4 & 2 & 4 \\ 0 & 6 & 7 & 6 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ 4×4

LOWER DIAGONAL / TRIANGULAR MATRIX

Definition: The matrix whose entries below the main diagonal are non-zero is upper diagonal.

$[8]$ 1×1 $\begin{bmatrix} 3 & 0 \\ 11 & 41 \end{bmatrix}$ 2×2 $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & 3 & 3 \end{bmatrix}$ 3×3 $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 5 & 7 & 4 & 0 \\ 1 & 6 & 7 & 2 \end{bmatrix}$ 4×4

SYMMETRIC MATRIX

Definition: The matrix whose entries above and below the main diagonal are equal is symmetric.

$[8]$ 1×1 $\begin{bmatrix} 3 & 11 \\ 11 & 41 \end{bmatrix}$ 2×2 $\begin{bmatrix} 2 & 1 & 7 \\ 1 & 4 & 3 \\ 7 & 3 & 3 \end{bmatrix}$ 3×3 $\begin{bmatrix} 2 & 3 & 5 & 1 \\ 3 & 6 & 7 & 6 \\ 5 & 7 & 4 & 7 \\ 1 & 6 & 7 & 2 \end{bmatrix}$ 4×4

IDENTITY MATRIX

Definition: A square matrix I in the RREF is called an identity matrix.

$$[1] \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

TRANSPOSE MATRIX

Definition: A transpose is a matrix in which the rows and columns are swapped.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \ 3 \times 2 \qquad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \ 2 \times 3 \qquad B = [1 \quad 2 \quad 3] \ 1 \times 3 \qquad B^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \ 3 \times 1$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \ 3 \times 3 \qquad C^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \ 3 \times 3 \qquad D = [5] \ 1 \times 1 \qquad D^T = [5] \ 1 \times 1$$

$$E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 3 \\ 3 & 5 & 1 & 2 \\ 6 & 3 & 2 & 1 \end{bmatrix} \ 4 \times 4 \qquad E^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 3 \\ 3 & 5 & 1 & 2 \\ 6 & 3 & 2 & 1 \end{bmatrix} \ 4 \times 4$$

Note: $E = E^T$ since E is symmetric.

MATRIX ADDITION / SUBTRACTION

Definition: Only matrices of the same size can be added or subtracted.

$$\begin{bmatrix} -4 & 2 & 1 \\ 5 & -33 & 11 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} \ \textit{does not exist}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix} + \begin{bmatrix} 4 & -4 \\ 5 & -5 \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 7 & -7 \\ 9 & -9 \end{bmatrix} \ 3 \times 2$$

$$\begin{bmatrix} -4 & 11 & 2 \\ 15 & 7 & -3 \\ -6 & 9 & 3 \end{bmatrix} - \begin{bmatrix} 11 & 4 & -1 \\ 1 & -2 & 3 \\ 5 & -3 & 33 \end{bmatrix} = \begin{bmatrix} -15 & 7 & 3 \\ 14 & 9 & -6 \\ -11 & 12 & -30 \end{bmatrix} \ 3 \times 3$$

$$\begin{bmatrix} 1 \\ -2 \\ 63 \\ 9 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 65 \\ 5 \end{bmatrix} \ 4 \times 1$$

MATRIX SCALAR MULTIPLICATION

Definition: The scalar is multiplied by each entry of the matrix.

$$k \cdot \begin{bmatrix} 9 & 8 & 7 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 9k & 8k & 7k \\ 4k & 5k & 6k \end{bmatrix} \qquad \frac{1}{3} \cdot \begin{bmatrix} 1 & 4 & 5 \\ -2 & 11 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{4}{3} & \frac{5}{3} \\ -\frac{2}{3} & \frac{11}{3} & \frac{7}{3} \end{bmatrix}$$

MATRIX MULTIPLICATION

Definition: Only matrices with equal inner sizes can be multiplied.

$$\begin{bmatrix} -1 & 2 \\ 3 & -7 \\ 4 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & 8 & -1 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -12 & 3 \\ -15 & 38 & -10 \\ -1 & 38 & -7 \end{bmatrix} \ 3 \times 3$$

$2 = 2$

$(3 \times 2) \times (2 \times 3) = 3 \times 3$

3×3

$$\begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix} \times \begin{bmatrix} 2 & -3 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & -12 \\ -12 & 18 \end{bmatrix} \ 3 \times 2$$

$1 = 1$

$(3 \times 1) \times (1 \times 2) = 3 \times 2$

3×2

$$\begin{bmatrix} 3 & -5 & 4 & 2 \\ -8 & 2 & 4 & -2 \\ 23 & -3 & 2 & 6 \\ 4 & 0 & -6 & 7 \end{bmatrix} \times \begin{bmatrix} 2 & -9 & 6 & 1 & 7 \\ -3 & 5 & 7 & 7 & 5 \\ 0 & -9 & 6 & 3 & 7 \\ -5 & 7 & 5 & 5 & 5 \end{bmatrix} \ \textit{does not exist}$$

$4 \neq 4$

$(4 \times 4) \times (4 \times 5)$

does not exist

MATRIX MULTIPLICATION PROPERTIES

Property 1: $A + B = B + A$ (commutative law for addition holds)

Property 2: $AB \neq BA$ (commutative law for multiplication fails)

$A = \begin{bmatrix} 3 & -3 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$ $\Rightarrow$ $AB = \begin{bmatrix} 3 & -21 \\ 11 & -3 \end{bmatrix}$, but $BA = \begin{bmatrix} 4 & -14 \\ 29 & -4 \end{bmatrix}$.

$A = \begin{bmatrix} 4 & -2 & 3 \\ -7 & 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $\Rightarrow$ $AB = \begin{bmatrix} 17 \\ 9 \end{bmatrix}$, but BA *does not exist*.

Property 3: $(kA)B = k(AB)$ (associative law for multiplication holds)

Property 4: A^k exists only if the matrix A is square.

Property 5: If a matrix is multiplied by an upper diagonal matrix, the resulting matrix will also be upper diagonal.

Property 6: If a matrix is multiplied by a lower diagonal matrix, the resulting matrix will also be lower diagonal.

ELEMENTARY MATRIX

Definition: A square matrix E is called elementary if it can be obtained from the corresponding identity matrix I in only one row operation.

$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is elementary since: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 = 2R_1 \quad \Leftrightarrow \quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is elementary since: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \textit{swap } R_1 \textit{ and } R_2 \quad \Leftrightarrow \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is elementary since: $I_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_2 = -1R_2 \quad \Leftrightarrow \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is elementary since: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 = R_2 + R_3 \quad \Leftrightarrow \quad D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is <u>not</u> elementary since it can be obtained from I_3 in two row operations.

$F = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is <u>not</u> elementary since it can be obtained from I_4 in two row operations.

$G = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ is <u>not</u> elementary because it is not square.

MULTIPLYING BY ELEMENTARY MATRICES

Definition: The result of multiplying by an elementary matrix is equivalent to one row operation.

1. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} R_1 = 2R_1 \Leftrightarrow \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \textit{swap } R_1 \textit{ and } R_2 \Leftrightarrow \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} R_2 = -1R_2 \Leftrightarrow \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 8 \end{bmatrix} \textit{swap } R_1 \textit{ and } R_4 \Leftrightarrow \begin{bmatrix} 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

INVERSE MATRIX

Definition: A square matrix A^{-1} is called the inverse of A if $AA^{-1} = I$.

Problem: Find the inverse of $A = \begin{bmatrix} -7 & -3 \\ 5 & 2 \end{bmatrix}$

Solution: We write A and I_2 side by side $\left(\begin{matrix} -7 & -3 \\ 5 & 2 \end{matrix}\middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$.

We have to convert A into RREF by applying the same row operations to both A and I_2.

$\left(\begin{matrix} -7 & -3 \\ 5 & 2 \end{matrix}\middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right) R_1 = 2R_1 + 3R_2 \quad \Leftrightarrow$

$\left(\begin{matrix} 1 & 0 \\ 5 & 2 \end{matrix}\middle| \begin{matrix} 2 & 3 \\ 0 & 1 \end{matrix}\right) R_2 = R_2 - 5R_1 \quad \Leftrightarrow$

$\left(\begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}\middle| \begin{matrix} 2 & 3 \\ -10 & -14 \end{matrix}\right) R_2 = R_2/2 \quad \Leftrightarrow$

$\left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle| \begin{matrix} 2 & 3 \\ -5 & -7 \end{matrix}\right) R_2 = R_2/2$

We have converted A into RREF, and on the right side we have converted I_2 to A^{-1}.

$\therefore A^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -7 \end{bmatrix}$ is the inverse of $A = \begin{bmatrix} -7 & -3 \\ 5 & 2 \end{bmatrix}$.

Check: We need to verify that $AA^{-1} = I$.

$\begin{bmatrix} -7 & -3 \\ 5 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ - the equality holds

$\therefore$ The inverse was computed correctly.

Problem: Find the inverse of $B = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$

Solution: Alternatively, we can use the following formula to compute B^{-1}.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Substituting B into the above formula yields:

$B^{-1} = \frac{1}{(-1)4-2(-3)} \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$.

Check: We need to verify that $BB^{-1} = I$.

$\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ - the equality holds.

$\therefore$ The inverse was computed correctly.

Problem: Find the inverse of $C = \begin{bmatrix} 5 & 16 & 45 \\ 3 & 10 & 30 \\ 4 & 13 & 38 \end{bmatrix}$

Solution: We write C and I_3 side by side $\left(\begin{array}{ccc|ccc} 5 & 16 & 45 & 1 & 0 & 0 \\ 3 & 10 & 30 & 0 & 1 & 0 \\ 4 & 13 & 38 & 0 & 0 & 1 \end{array}\right)$.

We have to convert A into RREF by applying the same row operations to both C and I_3.

$$\left(\begin{array}{ccc|ccc} 5 & 16 & 45 & 1 & 0 & 0 \\ 3 & 10 & 30 & 0 & 1 & 0 \\ 4 & 13 & 38 & 0 & 0 & 1 \end{array}\right) R_1 = R_1 - R_3 \quad \Leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & -1 \\ 3 & 10 & 30 & 0 & 1 & 0 \\ 4 & 13 & 38 & 0 & 0 & 1 \end{array}\right) R_2 = R_2 - 3R_1 \quad \Leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & -1 \\ 0 & 1 & 9 & -3 & 1 & 3 \\ 4 & 13 & 38 & 0 & 0 & 1 \end{array}\right) R_3 = R_3 - 4R_1 \quad \Leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & -1 \\ 0 & 1 & 9 & -3 & 1 & 3 \\ 0 & 1 & 10 & -4 & 0 & 5 \end{array}\right) R_3 = R_3 - R_2 \quad \Leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & -1 \\ 0 & 1 & 9 & -3 & 1 & 3 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right) R_2 = R_2 - 9R_3 \quad \Leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & -1 \\ 0 & 1 & 0 & 6 & 10 & -15 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right) R_1 = R_1 - 7R_3 \quad \Leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 8 & 7 & -15 \\ 0 & 1 & 0 & 6 & 10 & -15 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right) R_1 = R_1 - 3R_2 \quad \Leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & -23 & 30 \\ 0 & 1 & 0 & 6 & 10 & -15 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right)$$

We have converted C into RREF, and on the right side we have converted I_3 to C^{-1}.

$\therefore\ C^{-1} = \begin{bmatrix} -10 & -23 & 30 \\ 6 & 10 & -15 \\ -1 & -1 & 2 \end{bmatrix}$ is the inverse of $C = \begin{bmatrix} 5 & 16 & 45 \\ 3 & 10 & 30 \\ 4 & 13 & 38 \end{bmatrix}$.

Check: We need to verify that $CC^{-1} = I$.

$$\begin{bmatrix} 5 & 16 & 45 \\ 3 & 10 & 30 \\ 4 & 13 & 38 \end{bmatrix} \times \begin{bmatrix} -10 & -23 & 30 \\ 6 & 10 & -15 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ - the equality holds

$\therefore$ The inverse was computed correctly.

SOLVING SYSTEMS USING INVERSE MATRIX

Problem: Solve the system of equations $\begin{cases} 2x-4y+z=5 \\ 3x-3y+z=4 \\ -4x-2y-z=2 \end{cases}$ if the inverse of its augmented matrix is equal to $\begin{bmatrix} -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{9}{2} & -5 & -\frac{3}{2} \end{bmatrix}$.

Solution: Since $\begin{bmatrix} 2 & -4 & 1 \\ 3 & -3 & 1 \\ -4 & -2 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$, then $\begin{bmatrix} -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{9}{2} & -5 & -\frac{3}{2} \end{bmatrix} \times \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

All we need to do to find solutions is find the product of $\begin{bmatrix} -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{9}{2} & -5 & -\frac{3}{2} \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$.

$\begin{bmatrix} -\frac{5}{4} & \frac{3}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{9}{2} & -5 & -\frac{3}{2} \end{bmatrix} \times \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{5}{4} \\ -\frac{1}{2} \end{bmatrix}$, and so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{5}{4} \\ -\frac{1}{2} \end{bmatrix}$.

∴ The answer is $(1/4, -5/4, -1/2)$.

Problem: Solve the system of equations $\begin{cases} x+3y+z-7 \\ 2x+y+z=9 \\ 2x+2y+z=4 \end{cases}$ if the inverse of its augmented matrix is equal to $\begin{bmatrix} -1 & -1 & 2 \\ 0 & -1 & 1 \\ 2 & 4 & -5 \end{bmatrix}$.

Solution: Since $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 4 \end{bmatrix}$, then $\begin{bmatrix} -1 & -1 & 2 \\ 0 & -1 & 1 \\ 2 & 4 & -5 \end{bmatrix} \times \begin{bmatrix} -7 \\ 9 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

All we need to do to find solutions is find the product of $\begin{bmatrix} -1 & -1 & 2 \\ 0 & -1 & 1 \\ 2 & 4 & -5 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ 9 \\ 4 \end{bmatrix}$.

$\begin{bmatrix} -1 & -1 & 2 \\ 0 & -1 & 1 \\ 2 & 4 & -5 \end{bmatrix} \times \begin{bmatrix} -7 \\ 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 2 \end{bmatrix}$, and so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 2 \end{bmatrix}$.

∴ The answer is $(6, -5, 2)$.